Math 255A' Lecture 10 Notes

Daniel Raban

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1 Separation Results and Weak Topologies

1.1 Separation results in topological vector spaces

Last time, we had the geometric Hahn-Banach theorem.

Theorem 1.1. *Let X be a real TVS, and let G be a nonempty, open, convex subset with $x \in X \setminus G$. Then there exists*

- *an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f(x) = \alpha$ and $f(G) \subseteq (-\infty, \alpha)$,*
- *a closed affine hyperplane $M = \{f = \alpha\}$ such that $x \in M$ and $M \cap G = \varnothing$.*

This separates a point from a convex set. What about separating two convex sets?

Theorem 1.2. *Let X be a real TVS, and let A, B be disjoint convex sets with A open. Then there are an $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $A \subseteq \{f < \alpha\}$ and $B \subseteq \{f \geq \alpha\}$. If B is also open, then $B \subseteq \{f > \alpha\}$ (strict separation).*

Remark 1.1. This proof is difficult to imagine algebraically, but the main idea is only 1 step on top of the previous theorem.

Proof. Let $G := A - B = \{a - b : a \in A, b \in B\}$. This is convex, and we can also write $G = \bigcup_{b \in B}(A - b)$, which shows that G is open. Since $A \cap B = \varnothing$, $0 \notin G$. By the previous theorem, we find $f \in X^*$ such that $f[G] \subseteq (-\infty, 0)$. This set is $f[G] = f(A) - f(B)$. So $\alpha := \sup f[A] \leq \inf[B]$. Then $A \subseteq \{f \leq \alpha\}$ and $B \subseteq \{f \geq \alpha\}$. Because A is open, we can get $A \subseteq \{f < \alpha\}$. If B is open, we can do the same. $\square$

1.2 Separation results in locally convex spaces

Theorem 1.3. *Let X be a real LCS, and let A, B be disjoint, closed, convex subsets. If B is compact, they are strictly separated.*

Lemma 1.1. *Let $K \subseteq X$ be compact, and let $V \supseteq K$ be open. Then there is an open neighborhood $U \ni 0$ such that $K + U \subseteq V$.*

Proof. For each $x \in K$, there is a neighborhood U_x of 0 such that $x + Ux \subseteq V$. Because addition is continuous in X, there is a smaller neighborhood $W_x \ni 0$ such that $W_x - W_x \subseteq U$. Let $x \in K$, and suppose $x \in x_i + W_{x_i}$ By compactness, there exist $x_1, \ldots, x_n \in K$ such that $K \subseteq \bigcup_{i=1}^n (x_i + W_{x_i})$. Now take $W := \bigcap_{i=1}^n W_{x_i}$.

Let $x \in K$, and say $x \in x_i + W_{x_i}$. Then $x + W \subseteq x_i + W_{x_i} + W \subseteq x_i + W_{x_i} + W_{x_i} \subseteq V$. $\square$

Corollary 1.1. *If X is an LCS, we may take U to be convex.*

Proof. $B \subseteq X \setminus A$. The lemma gives a convex open $U \ni 0$ such taht $(B+U) \cap A = \varnothing$. The previous version of the theorem gives $f \in X^*$ and $\alpha \in \mathbb{R}$ such that $f[B] + f[U] \subseteq \{f < \alpha\}$ and $A \subseteq \{f \geq \alpha\}$. B is compact, so $f[B]$ is compact; so there exists some $\varepsilon > 0$ such that $f[B] \leq \alpha - \varepsilon$. Also, $f[A] \geq \alpha$. $\square$

Corollary 1.2. *Let X be a real LCS, let A be closed and convex, and let $x \in X \setminus A$. Then x, A are strictly separated.*

Corollary 1.3. *Let X be a real LCS, and let $A \subseteq X$. Then $\overline{\text{co}} A$ is the intersection of all closed half-spaces containing A.*

Corollary 1.4. *Let X be a real LCS, and let $A \subseteq X$. Then $\overline{\text{span}} A$ is the intersection of all closed hyperplanes containing A.*

Remark 1.2. These theorems all hold for complex vector spaces, as well. Here's how we get the complex vector space cases: If $f : X \to \mathbb{C}$. then let $f = \operatorname{Re} f$. Then $f = g(x) - ig(ix)$. In this case, when we say that two sets are separated, we mean the real part of f separates them.

1.3 Weak topologies

Example 1.1. Let (X, Σ, μ) be a finite measure space with no atoms (like the unit interval). Let $L^0(\mu)$ be the space of measurable functions $X \to \mathbb{C}$ with the topology of convergence in measure. So the topology is generated by sets of the form $\{f : \mu\{|f - g| > \varepsilon\} < \varepsilon\}$ for each $g \in L^0$.

There are no opne, convex sets besides the whole space. Assume U is convex with $U \ni 0$. Then $U \supseteq \{f : \mu\{|f| > \varepsilon\} < \varepsilon\}$. Let $n > 1/\varepsilon$. If $1 \leq i \leq n$, define $g_i = n(g\mathbb{1}_{[\frac{i-1}{n}, \frac{i}{n}]})$. Then $g = \frac{1}{n}(g_1, \ldots, g_n) \in U$.

Example 1.2. Let $C(\mathbb{R}^n)$ with the seminorms $p_K(f) := \|f|_K\|$ for all compact $K \subseteq \mathbb{R}^n$. Then if $L \in C(\mathbb{R}^N)^*$, then $|L| \leq \alpha p_K$ for some K. There exists a finite signed (or complex-valued) Borel measure $\mu \in M(K)$ such that $L(f) = \int f \, d\mu$ for all $f \in C(\mathbb{R}^n)$.

Let X be an LCS over $\mathbb{F}$.

Definition 1.1. If $x \in X$ and $x^* \in X^*$, we can write $\langle x, x^* \rangle$ as $x^*(x)$; we may also write $\langle x^*, x \rangle$.[1]

Definition 1.2. The **weak topology** on X is the topology generated by the seminorms $\{|f| : f \in X^*\}$. The **weak* topology** on X^* is the toplogy generated by $\{|\hat{x}| : x \in X\}$, where $\hat{x}(f) := f(x)$.

This is not stronger than the original topology.

Remark 1.3. Some authors refer to $\sigma(X, X^*)$ as the weak topology on X and $\sigma(X^*, X)$ as the weak* topology on X^*.

Theorem 1.4. *Let X be a locally convex space.*

1. $(X, \mathrm{wk})^* = X^*$.

2. $(X^*, \mathrm{wk}^*)^* = X$.

Lemma 1.2. *Let X be any vector space, and let $f, g_1, \ldots, g_n$ be linear functionals such that $\ker(f) \subseteq \bigcap_{i=1}^n \ker g_i$. Then $f \in \operatorname{span}\{g_1, \ldots, g_n\}$.*

Now let's prove the theorem.

Proof.

1. We need to check that $X^* \subseteq (X, \mathrm{wk})^*$. If $f \in X^*$, then $|f|$ is a generating seminorm for the weak topology on X, so $f \in (X, \mathrm{wk})^*$.

2. $(\subseteq)$: This is from the definition of wk*.

   $(\supseteq)$: Suppose $f : X^* \to \mathbb{F}$ is continuous for the wk* topology. Then there exist scalars $\alpha_1, \ldots, \alpha_n > 0$ and $x_1, \ldots, x_n$ such that $|f| \leq \sum_i \alpha_i |\hat{x}_i|$. Then $\ker(f) \subseteq \bigcap_{i=1}^n \ker \hat{x}_i$. The lemma then tells us that $f = \sum_{i=1}^n \beta_i \hat{x}_i = (\sum_{i=1}^n \beta_i x_i)^\wedge \in X$. □

For clarity, we will use the terms open, closed, and continuous to refer to the original toplogy on a space. We will use the terms weak-open, weak-closed, and weak-continuous to refer to the weak/weak* topology on a space.

[1] Conway's textbook says you can write it either way around because of some category theoretic duality. Professor Austin is pretty sure that it is because no one can remember which way it goes.